

Intrinsic Hölder continuity of harmonic functions

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Abstract

In a setting, where only “exit measures” are given, as they are associated with a right continuous strong Markov process on a separable metric space, we provide simple criteria for scaling invariant Hölder continuity of bounded harmonic functions with respect to a distance function which, in applications, may be adapted to the special situation. In particular, already a very weak scaling property ensures that Harnack inequalities imply Hölder continuity. Our approach covers recent results by M. Kassmann and A. Mimica as well as cases, where a Green function leads to an intrinsic metric.

Keywords: Harmonic function; Hölder continuity; right process; balayage space; Lévy process

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1 Harmonic functions in a general setting

During the last years, Hölder continuity of bounded harmonic functions has been studied for various classes of Markov processes (see [14, 8] and the references therein). The aim of this paper is to offer not only a unified (and perhaps more transparent) approach to results obtained until now, but also the possibility for applications in new cases.

Let X be a topological space such that finite measures μ on its σ -algebra $\mathcal{B}(X)$ of Borel subsets satisfy

$$\mu(A) = \sup\{\mu(F) : F \text{ closed}, F \subset A\}, \quad A \in \mathcal{B}(X).$$

This holds if X is a separable metric space (on its completion every finite measure is tight). Let $\mathcal{M}(X)$ denote the set of all finite measures on $(X, \mathcal{B}(X))$ (which we also consider as measures on the σ -algebra $\mathcal{B}^*(X)$ of all universally measurable sets). Given a set \mathcal{F} of numerical functions on X , let $\mathcal{F}_b, \mathcal{F}^+$ be the set of all functions in \mathcal{F} which are bounded, positive respectively.

For great flexibility in applications, we consider an open neighborhood X_0 of a point $x_0 \in X$ and suppose that we have a continuous real function $\rho_0 \geq 0$ on X_0 with $\rho_0(x_0) = 0$ and $0 < R_0 \leq \infty$ such that, for every $0 < r < R_0$, the closure of

$$U_r := \{x \in X : \rho_0(x) < r\}$$

is contained in X_0 . Let \mathcal{U}_0 denote the set of all open sets V in X with $V \subset U_r$ for some $0 < r < R_0$.

We suppose that we have measures $\mu_x^U \in \mathcal{M}(X)$, $x \in X$, $U \in \mathcal{U}_0$, such that the following hold for all $x \in X$ and $U, V \in \mathcal{U}_0$ (where ε_x is the Dirac measure at x):

- (i) The measure μ_x^U is supported by U^c , and $\mu_x^U(X) = 1$. If $x \in U^c$, then $\mu_x^U = \varepsilon_x$.
- (ii) The functions $y \mapsto \mu_y^U(E)$, $E \in \mathcal{B}(X)$, are universally measurable on X and

$$(1.1) \quad \mu_x^U = (\mu_x^V)^U := \int \mu_y^U d\mu_x^V(y), \quad \text{if } V \subset U.$$

Of course, stochastic processes and potential theory abundantly provide examples (with $X_0 = X$, $\rho_0(x) = \rho(x, x_0)$, where ρ is *any* metric for the topology of X).

EXAMPLES 1.1. 1. Right process \mathfrak{X} with strong Markov property on a Radon space X ,

$$\tau_U := \inf\{t \geq 0: X_t \in U^c\} < \infty \text{ } \mathbb{P}^x\text{-a.s.} \quad \text{and} \quad \mu_x^U(E) := P^x[X_{\tau_U} \in E]$$

for all $U \in \mathcal{U}_0$, $x \in X$, $E \in \mathcal{B}(X)$ ([3, Propositions 1.6.5 and 1.7.11, Theorem 1.8.5]).

If $U, V \in \mathcal{U}_0$ with $V \subset U$, then $\tau_U = \tau_V + \tau_U \circ \theta_{\tau_V}$, and hence, by the strong Markov property, for all $x \in X$ and $E \in \mathcal{B}(X)$,

$$\mu_x^U(E) = \mathbb{P}^x[X_{\tau_U} \in E] = \mathbb{E}^x(\mathbb{P}^{X_{\tau_V}}[X_{\tau_U} \in E]) = \int \mu_y^U(E) d\mu_x^V(y).$$

2. Balayage space (X, \mathcal{W}) (see [4]) such that $1 \in \mathcal{W}$,

$$\int v d\mu_x^U = R_v^{U^c}(x) := \inf\{w(x): w \in \mathcal{W}, w \geq v \text{ on } U^c\}, \quad v \in \mathcal{W}.$$

The properties (i) and (ii) follow from [4, VI.2.1, 2.4, 2.10, 9.1].

Given $U \in \mathcal{U}_0$, let $\mathcal{H}(U)$ denote the set of all universally measurable real functions h on X which are *harmonic on U* , that is, for all open sets V with $\overline{V} \subset U$ and all $x \in V$, are μ_x^V -integrable and satisfy

$$(1.2) \quad \int h d\mu_x^V = h(x).$$

Obviously, constant functions are harmonic on U and, for every bounded Borel measurable function f on X , the function $x \mapsto \int f d\mu_x^U$ is harmonic on U , by (1.1). The latter even holds for every bounded *universally* measurable function f on X (see [10, Section 2]).

2 Main result

Our aim is to obtain criteria for the following scaling invariant Hölder continuity of bounded harmonic functions.

(HC) There exist $C > 0$ and $\beta \in (0, 1)$ such that, for all $0 < r < R_0$,

$$|h(x) - h(x_0)| \leq C \|h\|_\infty \left(\frac{\rho_0(x)}{r} \right)^\beta \quad \text{for all } h \in \mathcal{H}_b(U_r) \text{ and } x \in U_r.$$

To that end we introduce the following properties.

(J₁) There are $\alpha, \delta_0 \in (0, 1)$ such that, for every $0 < r < R_0$ and every universally measurable set A in U_r ,

$$\inf\{\mu_x^{U_{\alpha r}}(A) : x \in U_{\alpha^2 r}\} > \delta_0 \quad \text{or} \quad \inf\{\mu_x^{U_{\alpha r}}(U_r \setminus A) : x \in U_{\alpha^2 r}\} > \delta_0.$$

(J₂) There are $\alpha_0, a_0 \in (0, 1)$ and $C_0 \geq 1$ such that, for all $0 < r < R_0$ and $n \in \mathbb{N}$,

$$(2.1) \quad \mu_x^{U_{\alpha_0^n r}}(U_r^c) \leq C_0 a_0^n \quad \text{for every } x \in U_{\alpha_0^{n+1} r}.$$

Of course, (J₂) holds trivially if the harmonic measures μ_x^U , $U \in \mathcal{U}_0$, $x \in U$, are supported by the boundary of U , that is, in the Examples 1.1, if \mathfrak{X} is a diffusion or (X, \mathcal{W}) is a harmonic space (since then $\mu_x^{U_{\alpha^n r}}(U_r^c) = 0$ for $x \in U_{\alpha^n r}$).

LEMMA 2.1. *If (J₂) holds, then, for every $a \in (0, 1)$, there exists $\alpha \in (0, 1)$ such that, for all $0 < r < R_0$ and $n \in \mathbb{N}$,*

$$(2.2) \quad \mu_x^{U_{\alpha^n r}}(U_r^c) \leq a^n \quad \text{for every } x \in U_{\alpha^{n+1} r}.$$

Proof. If (J₂) holds with α_0, a_0, C_0 , we fix $k \in \mathbb{N}$ with $a_0^k < C_0^{-1}a$, and take $\alpha := \alpha_0^k$. Then, for all $n \in \mathbb{N}$ and $x \in U_{\alpha^{n+1} r} \subset U_{\alpha_0^{kn+1} r}$,

$$\mu_x^{U_{\alpha^n r}}(U_r^c) \leq C_0 a_0^{kn} \leq C_0 (C_0^{-1}a)^n \leq a^n.$$

□

REMARK 2.2. If (J₁), (J₂) or (2.2) hold for some $\alpha \in (0, 1)$, then, by (1.1), they hold for any smaller α (keeping the other constants).

THEOREM 2.3. *Suppose (J₁) and (J₂). Then (HC) holds (with C and β which depend only on the constants in (J₁) and (J₂)).*

Proof (cf. the proofs of [2, Theorem 4.1] and [14, Theorem 1.4]). Let $\delta_0 \in (0, 1)$ satisfy (J₁). We define $\delta := \delta_0/6$, $a := \delta/2$. Then $a < (1 - a)\delta$, and there exists $1 < b < \sqrt{3/2}$ with

$$(2.3) \quad b^3(1 - 3\delta) \leq 1 - 2\delta \quad \text{and} \quad ab^4 < (1 - ab)\delta.$$

By Lemma 2.1 and Remark 2.2, there exists $\alpha \in (0, 1)$ such that the statement in (J₁) holds with this α and δ_0 , and

$$(2.4) \quad \mu_x^{U_{\alpha^k r}}(U_r^c) \leq a^k, \quad 0 < r < R_0, k \in \mathbb{N}, x \in U_{\alpha^{k+1} r}.$$

We now fix $0 < r < R_0$ and $h \in \mathcal{H}_b(U_r)$ with $\|h\|_\infty = 1$. For $n = 0, 1, 2, \dots$ let

$$B_n := U_{\alpha^n r}, \quad M_n(h) := \frac{1}{2}(\sup h(B_n) + \inf h(B_n)),$$

$$\text{osc}_n(h) := \sup h(B_n) - \inf h(B_n) = \sup\{h(x) - h(y) : x, y \in B_n\},$$

Clearly, $M_n(-h) = -M_n(h)$ and $\text{osc}_n(-h) = \text{osc}_n(h)$. We claim that, for $n \geq 0$,

$$(2.5) \quad \text{osc}_n(h) \leq s_n := 3b^{-n}.$$

Clearly, (2.5) holds trivially for $n = 0, 1, 2$, since $\text{osc}_n(h) \leq 2$ and $b^2 < 3/2$.

Let us consider $n \geq 2$, suppose that (2.5) holds for all $0 \leq j \leq n$, and define

$$A(h) := \{z \in B_n : h(z) \leq M_n(h)\}, \quad B(h) := \{z \in B_n : h(z) \geq M_n(h)\}.$$

By (J₁),

$$\inf\{\mu_x^{B_{n+1}}(A(h)) : x \in B_{n+2}\} > \delta_0 \quad \text{or} \quad \inf\{\mu_x^{B_{n+1}}(B(h)) : x \in B_{n+2}\} > \delta_0.$$

Without loss of generality $\inf\{\mu_x^{B_{n+1}}(A(h)) : x \in B_{n+2}\} > \delta_0$ (otherwise we replace h by $-h$ using $B(h) = A(-h)$).

Now let us fix two points $x, y \in B_{n+2}$. We claim that

$$(2.6) \quad h(x) - h(y) \leq s_{n+1}.$$

We may choose a closed set F in $A(h)$ such that $\mu_x^{B_{n+1}}(F) > \delta_0$. Then

$$\mu := \mu_x^{B_n \setminus F}$$

satisfies $\mu(F) > \delta_0 = 6\delta$, by (1.1). Since $h \in \mathcal{H}_b(U_r)$ and μ is a probability measure,

$$(2.7) \quad h(x) - h(y) = \int (h - h(y)) d\mu.$$

The measure μ is supported by $F \cup B_n^c$. Clearly,

$$\int_F (h - h(y)) d\mu \leq (M_n(h) - \inf h(B_n))\mu(F) = \frac{1}{2} \text{osc}_n(h)\mu(F) < \frac{1}{2} s_{n-1} \mu(F).$$

Since $\mu(B_n^c) = 1 - \mu(F)$, we see that

$$\int_{B_{n-1} \setminus B_n} (h - h(y)) d\mu \leq s_{n-1}(1 - \mu(F)).$$

Combining the two previous estimates we obtain that

$$(2.8) \quad \int_{F \cup (B_{n-1} \setminus B_n)} (h - h(y)) d\mu \leq s_{n-1}(1 - \frac{1}{2}\mu(F)) \leq s_{n+2}(1 - 2\delta),$$

where the last inequality follows from $\mu(F) > 6\delta$ and (2.3). Moreover

$$(2.9) \quad \int_{B_{n-1}^c} (h - h(y)) d\mu \leq 2\mu(B_0^c) + \sum_{j=0}^{n-2} s_j \mu(B_j \setminus B_{j+1}).$$

By (1.1) and (2.4) (applied to $\alpha^m r$ in place of r), for all $0 \leq m \leq n$,

$$\mu(B_m^c) \leq \mu_x^{B_n}(B_m^c) \leq a^{n-m}.$$

By (2.3), $ab^4 < (1 - ab)\delta$. Hence $2\mu(B_0^c) \leq 2a^n \leq 2(b^{-4}\delta)^n < s_{n+2}\delta$ and

$$\sum_{j=0}^{n-2} s_j \mu(B_j \setminus B_{j+1}) \leq \sum_{j=0}^{n-2} s_j \mu(B_{j+1}^c) \leq 3 \sum_{j=0}^{n-2} b^{-j} a^{n-(j+1)} = s_{n+2}s,$$

where

$$s = b^{n+2} \sum_{j=0}^{n-2} b^{-j} a^{n-(j+1)} = b^3 \sum_{j=0}^{n-2} (ab)^{n-(j+1)} < \frac{ab^4}{1 - ab} < \delta.$$

Having (2.7), the estimates (2.8) and (2.9) hence yield $h(x) - h(y) \leq s_{n+2}$. So (2.6) holds, and we see that $\text{osc}_{n+2}(h) \leq s_{n+2}$.

Given $x \in B_0 \setminus \{x_0\}$, there exists $n \geq 0$ such that $x \in B_n \setminus B_{n+1}$. Defining $\beta := (\ln b)/\ln(1/\alpha)$, we finally conclude that

$$|h(x) - h(x_0)| \leq 3b^{-n} = 3\alpha^{n\beta} \leq 3\left(\frac{\rho_0(x)}{\alpha r}\right)^\beta.$$

□

REMARK 2.4. The proof shows that dealing with harmonic functions which are Borel measurable, continuous, respectively, we need (J₁) only for sets A in U_r which are Borel measurable, relatively closed in U_r , respectively.

To see that (J₁) is almost necessary for Hölder continuity of bounded harmonic functions we introduce the following weak property which immediately follows from both (J₁) and (J₂) and merely states that ρ_0 provides a suitable scaling at x_0 .

(J₀) There are $\alpha, \delta_0 \in (0, 1)$ such that, for every $0 < r < R_0$,

$$(2.10) \quad \mu_{x_0}^{U_{\alpha r}}(U_r) > \delta_0.$$

For the moment, let us fix $0 < r < R_0$, $\alpha \in (0, 1)$, let $S := U_r \setminus U_{\alpha r}$ and A be a universally measurable set in X . Of course, $\mu_{x_0}^{U_{\alpha r}}(U_r) = \mu_{x_0}^{U_{\alpha r}}(S)$, hence (2.10) implies that $\mu_{x_0}^{U_{\alpha r}}(S \cap A) > \delta_0/2$ or $\mu_{x_0}^{U_{\alpha r}}(S \setminus A) > \delta_0/2$, and there exists a closed set F in $S \cap A$ or $S \setminus A$ such that

$$(2.11) \quad \mu_{x_0}^{U_{\alpha r}}(F) > \delta_0/2.$$

PROPOSITION 2.5. *Assuming (J₀), property (J₁) is necessary for (HC).*

Proof. Suppose that (HC) and (J₀) hold with $C, \beta, \alpha_0, \delta_0$. We choose $0 < \alpha < \alpha_0$ such that $C\alpha^\beta < \delta_0/4$. Let $0 < r < R_0$ and F be a closed set in $U_r \setminus U_{\alpha r}$ such that (2.11) holds. The function $x \mapsto \mu_x^{U_{\alpha r}}(F)$ is harmonic on $U_{\alpha r}$. So (HC) implies that, for every $x \in U_{\alpha^2 r}$,

$$\mu_x^{U_{\alpha r}}(F) \geq \mu_{x_0}^{U_{\alpha r}}(F) - C\alpha^\beta > \delta_0/4.$$

□

COROLLARY 2.6. *Suppose that the measures μ_x^U , $x \in U \in \mathcal{U}_0$, are supported by the boundary of U . Then (HC) holds if and only if (J₁) holds.*

Moreover, Theorem 2.3 will quickly lead to Corollary 3.2 which, in turn, yields Hölder continuity of bounded harmonic functions and continuity of harmonic functions provided there is a suitable associated Green function (see Remark 3.3). For a direct application of Theorem 2.3 in the setting of [14] see Sections 4 and 5.

3 Harnack inequalities imply Hölder continuity

Next let us see that the following scaling invariant Harnack inequalities are sufficient for (HC) provided (J₀) holds.

(HI) Harnack inequality: There exist $\alpha \in (0, 1)$ and $K \geq 1$ such that

$$\sup h(U_{\alpha r}) \leq K \inf h(U_{\alpha r}) \quad \text{for all } 0 < r < R_0 \text{ and } h \in \mathcal{H}_b^+(U_r).$$

PROPOSITION 3.1. *If (J₀) and (HI) hold, then (J₁) and (J₂) are satisfied.*

Proof. Let $\alpha, \delta_0 \in (0, 1)$ satisfy (J₀) and (HI), and let $0 < r < R_0$.

1) Let F be a closed set in $U_r \setminus U_{\alpha r}$ such that (2.11) holds. By (HI), the harmonicity of the function $x \mapsto \mu_x^{U_{\alpha r}}(F)$ on $U_{\alpha r}$ yields that $\mu_x^{U_{\alpha r}}(F) > (2K)^{-1}\delta_0$ for every $x \in U_{\alpha^2 r}$. So (J₁) holds (with $(2K)^{-1}\delta_0$ in place of δ_0).

2) Let $0 < s \leq r$. By (J₀), there exists a closed set F in $U_{\alpha s} \setminus U_{\alpha^2 s}$ such that $\mu_{x_0}^{U_{\alpha^2 s}}(F) > \delta_0$. Let us fix $x \in U_{\alpha^3 r}$. By (HI) and (1.1),

$$K \mu_x^{U_s \setminus F}(F) \geq \mu_{x_0}^{U_s \setminus F}(F) \geq \mu_{x_0}^{U_{\alpha^2 s}}(F) > \delta_0.$$

By (1.1),

$$\mu_x^{U_{\alpha^2 s}}(U_r^c) \leq \mu_x^{U_s \setminus F}(U_r^c) = \mu_x^{U_s}(U_r^c) - \int_F \mu_y^{U_s}(U_r^c) d\mu_x^{U_s \setminus F},$$

where, for every $y \in F$, $\mu_y^{U_s}(U_r^c) \geq K^{-1}\mu_x^{U_s}(U_r^c)$, by (HI). Therefore

$$\mu_x^{U_{\alpha^2 s}}(U_r^c) \leq (1 - K^{-1}\mu_x^{U_s \setminus F}(F))\mu_x^{U_s}(U_r^c) \leq (1 - K^{-2}\delta_0)\mu_x^{U_s}(U_r^c).$$

Proceeding by induction, we get (J₂) with $a := 1 - K^{-2}\delta_0$, $C_0 = 1$ (since $\mu_x^{U_r}(U_r^c) = 1$) and α^2 in place of α . \square

Thus Theorem 2.3 leads to the following result (where we might recall that (J₀) trivially holds if, for every $0 < r < R_0$, the measure $\mu_{x_0}^{U_r}$ is supported by ∂U_r).

COROLLARY 3.2. *(J₀) and (HI) imply (HC).*

REMARK 3.3. For applications, where properties of an associated Green function imply (HI), see [10, Theorems 4.12, 5.2, 6.2, 6.3 and 7.3].

4 A general application using the Dynkin formula and the Lévy system formula

In this section we shall present a consequence of Theorem 2.3 which can immediately be applied to the setting considered in [14] (see Section 5).

Let $X = \mathbb{R}^d$, $d \geq 1$, and, for $x_0 \in \mathbb{R}^d$ and $0 < r \leq \infty$, let

$$B(x_0, r) := \{x \in \mathbb{R}^d : |x - x_0| < r\}, \quad B_r := B(0, r).$$

Let us fix $K_0, c_0, c_1, c_3 \in (1, \infty)$. Further, let $0 < R < R_0 \leq \infty$ and $U_0 := B_{2R_0}$. We assume that we have a measurable function $K : U_0 \times \mathbb{R}^d \rightarrow [0, \infty)$ and a continuous function $l : (0, R_0) \rightarrow (0, \infty)$ such that the following hold.

(K) For all $x \in B_{2R}$ and $h \in B_1$, $K(x, h) = K(x, -h)$, and

$$\sup_{x \in U_0} \int_{\mathbb{R}^d} (1 \wedge |h|^2) K(x, h) dh \leq K_0.$$

(L₀) For all $x \in U_0$ and $h \in \mathbb{R}^d$ with $|h| < R_0$,

$$c_0^{-1} |h|^{-d} l(|h|) \leq K(x, h) \leq c_0 |h|^{-d} l(|h|).$$

(L₁) For all $0 < r \leq s < R$,

$$l(r/2) \leq c_1 l(r) \quad \text{and} \quad s^{-d} l(s) \leq c_1 r^{-d} l(r).$$

(L₂) Defining $L(r) := \int_r^{R_0} u^{-1} l(u) du$, $0 < r \leq R_0$, we have $L(0) = \infty$ and

$$\tilde{L}(r) := r^{-2} \int_0^r ul(u) du \leq c_2 L(r) \quad \text{for every } 0 < r < R.$$

(L₃) $L(R/2) + (1 \vee R^{-2}) K_0 \leq c_3 L(R)$.

Moreover, we suppose that there exists a strong Markov process $\mathfrak{X} = (X_t, \mathbb{P}^x)$ on \mathbb{R}^d with trajectories that are right continuous and have left limits and such that, for all $x_0 \in B_R$, $0 < r < R$ and $x \in B(x_0, r)$, the following holds for every $t > 0$ and

$$\tau_r := \inf\{u \geq 0 : X_u \notin B(x_0, r)\}.$$

(D) Dynkin formula: For all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ with compact support,

$$\mathbb{E}^x f(X_{\tau_r \wedge t}) - f(x) = \mathbb{E}^x \int_0^{\tau_r \wedge t} \int_{\mathbb{R}^d} (f(X_u + h) - f(X_u)) K(X_u, h) dh du.^1$$

(LS) Lévy system formula: For all Borel sets A in $B(x_0, r)^c$,

$$\mathbb{P}^x[X_{\tau_r \wedge t} \in A] = \mathbb{E}^x \int_0^{\tau_r \wedge t} \int_A K(X_u, z - X_u) dz du.$$

The existence of such a process is assured if $K(x, h)$ does not depend on x ; in the general case it has been established in various contexts (see the discussion in [1]).

The only reason for assuming the weird condition (L₃) is that we then may stress that constants β , C and $C_j \in (1, \infty)$, $1 \leq j \leq 5$, introduced later on, are valid for all R_0 , R , K and l satisfying (K), (L₀) – (L₃), (D) and (LS).

Let us observe right away that $\int_{\{r < |h| < R_0\}} K(x_0, h) dh \approx L(r)$, by (L₀), and hence (K) implies that $0 < L(r) < \infty$ on $(0, R_0)$, and L is strictly decreasing and continuous on $[0, R_0]$ (with $L(R_0) = 0$).

We claim that Theorem 2.3 leads to a result, which immediately implies the statement of Theorem 3 in the case $f = 0$ and Theorem 12 in [14] (see Section 5 and Corollary 4.7).

¹A term $\langle \nabla f(X_u), h \rangle \mathbb{1}_{\{|h| < 1\}} K(X_u, h)$ the reader may expect in the integral on \mathbb{R}^d does not yield any contribution because of $K(\cdot, h) = K(\cdot, -h)$.

THEOREM 4.1. *There exist $C > 0$ and $\beta \in (0, 1)$ such that, for all $x_0 \in B_R$, $0 < r < R$, $h \in \mathcal{H}_b(B(x_0, r))$ and $x \in B(x_0, r)$,*

$$|h(x) - h(x_0)| \leq C \|h\|_\infty \left(\frac{L(|x - x_0|)}{L(r)} \right)^{-\beta}.$$

The proof of our claim will be based on the next proposition which essentially consists of rearranged results from [14, Section 6].

Let $x_0 \in B_R$ and, for every $r > 0$,

$$V_r := B(x_0, r) \quad \text{and} \quad \tau_r := \inf\{u \geq 0 : X_u \notin V_r\}.$$

Moreover, let κ_d denote the surface measure of B_1 , let $k(u) := u^{-d}l(u)$ and let μ be the measure on V_{R_0} having density $k(|x - x_0|)/L(|x - x_0|)$ with respect to Lebesgue measure.

PROPOSITION 4.2. *There are $C_1, C_2, C_3 \in (1, \infty)$ such that the following holds.*

(1) *Let $0 < r < R$ and $x \in V_r$. Then*

$$(4.1) \quad E^x \tau_r \leq C_1 L(r)^{-1}.$$

If $r < s < R$ and $a := L(r)/L(s)$, then, for every Borel set A in $S := V_s \setminus V_r$,

$$(4.2) \quad \mathbb{P}^x[X_{\tau_r} \in A] \geq C_1^{-1} \frac{\ln a}{a} \frac{\mu(A)}{\mu(S)} L(r) E^x \tau_r.$$

(2) *If $r, s \in (0, R)$ such that $r \leq s/2$, then, for every $x \in V_r$,*

$$\mathbb{P}^x[X_{\tau_r} \notin V_s] \leq C_2 L(s) E^x \tau_r.$$

(3) *If $0 < r < R$ and $x \in V_{r/2}$, then $E^x \tau_r \geq C_3^{-1} L(r)^{-1}$.*

Let us note that (1) states what can be obtained from the proof of [14, Proposition 17], which is based on (LS). Hence a separate proof of the first part of [14, Proposition 15], giving an upper estimate of $\mathbb{E}^x \tau_r$ by a multiple of $L(r)^{-1}$, is not needed. Moreover, (2) is [14, Proposition 16] (having a proof using (LS) and the upper estimate for $\mathbb{E}^x \tau_r$). Finally, a modification of the proof (the only one using (L₂) and (D)) given in [14, Proposition 14] for an upper estimate of $t^{-1} \mathbb{P}[\tau_r \leq t]$ by a multiple of $L(r)$ directly yields (3) so that also the second part of [14, Proposition 15] is obtained without an additional proof.

Because of these simplifications let us write down a complete proof for Proposition 4.2. We first establish two simple facts which are repeatedly used also in [14].

LEMMA 4.3. *Let $C_4 := 1 + c_1 + c_3$. Then $L(r/2) \leq C_4 L(r)$ for every $0 < r < R$.*

Proof. Let us first consider $0 < r \leq R/2$. Then $L(r/2) = L(r) + I_r$, where, by (L₁),

$$I_r = \int_{r/2}^r u^{-1} l(u) du \leq c_1 \int_{r/2}^r u^{-1} l(2u) du = c_1 \int_r^{2r} v^{-1} l(v) dv \leq c_1 L(r).$$

If $R/2 < r < R$, then $L(r) < L(R/2) \leq c_3 L(R) < c_3 L(r)$, by (L₂). □

LEMMA 4.4. For all $x \in U_0$ and $0 < r < R$, $\int_{\{|h|>1\wedge R\}} K(x, h) dh \leq c_3 L(r)$.

Proof. Let $a := 1 \vee R^{-2}$. By (K) and (L₃),

$$\int_{\{|h|>1\wedge R\}} K(x, h) dh \leq a \int_{\mathbb{R}^d} (1 \wedge |h|^2) K(x, h) dh \leq a K_0 \leq c_3 L(R).$$

for every $x \in U_0$. It remains to observe that $L(R) < L$ on $(0, R)$. \square

Proof of Proposition 4.2. (1) Let A be a Borel set in V_r^c . If $y \in V_r$ and $z \in V_r^c$, then $|z - y| \leq |z - x_0| + r \leq 2|z - x_0|$, hence $k(|z - x_0|) \leq c_1 k(|z - y|/2) \leq 2^d c_1^2 k(|z - y|)$, by (L₁). So (LS) implies that, for $t > 0$,

$$\begin{aligned} 1 &\geq \mathbb{P}^x[X_{\tau_r \wedge t} \in A] = \mathbb{E}^x \int_0^{\tau_r \wedge t} \int_A K(X_u, z - X_u) dz du \\ &\geq c_0^{-1} \mathbb{E}^x \int_0^{\tau_r \wedge t} \int_A k(|z - X_u|) dz du \geq (2^d c_0 c_1^2)^{-1} \mathbb{E}^x(\tau_r \wedge t) \int_A k(|z - x_0|) dz. \end{aligned}$$

Since $\int_{V_{R_0} \setminus V_r} k(|z - x_0|) dz = \kappa_d L(r)$, (4.1) follows taking $C_1 := 2^d \kappa_d^{-1} c_0 c_1^2$ and letting t tend to infinity.

Now let $r < s < R_0$ and $a := L(r)/L(s)$. Then,

$$(4.3) \quad \kappa_d^{-1} \mu(S) = \int_r^s \frac{l(u)}{uL(u)} du = -\ln L(u)|_r^s = \ln a,$$

and hence, if $A \subset S$,

$$\int_A k(|z - x_0|) dz = \int_A L(|z - x_0|) d\mu(z) \geq L(s) \mu(A) = \kappa_d \frac{\ln a}{a} L(r) \frac{\mu(A)}{\mu(S)}.$$

(2) Let $0 < 2r \leq s < R$. By (LS) (recall that $\tau_r < \infty$ \mathbb{P}^x -a.s. by (1)),

$$\mathbb{P}^x[X_{\tau_r} \in V_s^c] = \mathbb{E}^x \int_0^{\tau_r} \int_{V_s^c} K(X_u, z - X_u) dz du.$$

If $y \in V_r$, then $B(y, s/4) \subset V_{3s/4}$. Hence, by Lemmas 4.3 and 4.4,

$$\int_{V_s^c} K(y, z - y) dz \leq \int_{\{s/4 < |h|\}} K(y, h) dh \leq \kappa_d c_0 L(s/4) + c_3 \leq C_2 L(s),$$

where $C_2 := \kappa_d c_0 C_4^2 + c_3$. Thus $\mathbb{P}^x[X_{\tau_r} \in V_s^c] \leq C_2 L(s) \mathbb{E}^x \tau_r$.

(3) Let $\psi(u) \in \mathcal{C}^\infty(\mathbb{R})$ such that $\psi(u) = u^2 - 5$ for every $u \in [-2, 2]$, $\psi = 0$ on $\mathbb{R} \setminus [-3, 3]$ and $-5 \leq \psi \leq 0$. Let $0 < r < R$, $s := 1 \wedge r$ and, for $y, h \in \mathbb{R}^d$,

$$f(y) := \psi(|y - x_0|/r) \quad \text{and} \quad F(y, h) := (f(y) - f(y + h))K(y, h).$$

By (D), for every $x \in V_r$ and $t > 0$,

$$(4.4) \quad \mathbb{E}^x f(X_{\tau_r \wedge t}) - f(x) = \mathbb{E}^x \int_0^{\tau_r \wedge t} \int_{\mathbb{R}^d} F(X_u, h) dh du.$$

Let $y \in V_r$. Since $-5 \leq f \leq 0$, we have $|F| \leq 5K$, and hence, by (L₀) and Lemma 4.4,

$$\int_{\{s < |h|\}} |F(y, h)| dh \leq 5 \int_{\{r < |h| < R_0\} \cup \{1 \wedge R < |h|\}} K(y, h) dh \leq 5(\kappa_d c_0 + c_3)L(r).$$

By (K₀), $K(y, h) = K(y, -h)$ for every $y \in B_1$. Hence, by (L₀) and (L₂),

$$\begin{aligned} (4.5) \quad \int_{\{|h| < s\}} F(y, h) dh &= \int_{\{|h| < s\}} (f(y+h) - f(y) - \langle \nabla f(y), h \rangle) K(y, h) dh \\ &= \int_{\{|h| < s\}} \frac{|h|^2}{r^2} K(y, h) dh \leq \kappa_d c_0 r^{-2} \int_0^r ul(u) du \leq \kappa_d c_0 c_2 L(r). \end{aligned}$$

Combining the preceding estimates we see that, defining $C_3 := 10\kappa_d c_0(1 + c_2) + c_3$,

$$(4.6) \quad \int_{\mathbb{R}^d} F(y, h) dh \leq (1/2)C_3 L(r).$$

Finally, let $x \in V_{r/2}$. Then, by (4.4) and (4.6), for every $t > 0$,

$$\mathbb{E}^x f(X_{\tau_r \wedge t}) - f(x) \leq (1/2)C_3 L(r) \mathbb{E}^x(\tau_r),$$

where $f(x) - f(X_{\tau_r}) > 1/2$ on $\{\tau_r < \infty\}$. Letting $t \rightarrow \infty$ we hence see that

$$1/2 < \mathbb{E}^x f(X_{\tau_r}) - f(x) \leq (1/2)C_3 L(r) \mathbb{E}^x(\tau_r)$$

completing the proof. \square

REMARK 4.5. Suppose for a moment that instead of having (L₂), which is equivalent to $\limsup_{r \rightarrow 0} \tilde{L}(r)/L(r) < \infty$, we would have $\lim_{r \rightarrow 0} \tilde{L}(r)/L(r) = \infty$, the preceding proof could easily be modified (using the equalities in (4.5) for $r < 1 \wedge R$) to show that then $E^x \tau_r \approx \tilde{L}(r)^{-1}$ for $0 < r < R$ and $x \in V_{r/2}$. This is the case, if $l(u) = u^{-2}(\ln u^{-1})^{-2}$ (see the end of Section 5).

To apply Theorem 2.3, we define

$$\rho_0(x) := L(|x - x_0|)^{-1}, \quad x \in X_0 := V_R.$$

Then, for every $0 < r < R$,

$$V_r = U_{L(r)^{-1}}.$$

COROLLARY 4.6. *There exist $\alpha, a_0, \delta_0 \in (0, 1)$ and $C_0 \geq 1$, which depend only on K_0, c_0, c_1, c_2, c_3 , satisfying (J₁) and (J₂).*

Proof. If $r, s \in (0, R)$, then

$$(4.7) \quad r < s/2 \quad \text{provided} \quad L(r) > C_4 L(s),$$

since L is strictly decreasing and $C_4 L(s) \geq L(s/2)$, by Lemma 4.3. We define

$$\alpha := a_0 := C_4^{-1}, \quad \delta_0 := (3C_1 C_3 C_4)^{-1}, \quad C_0 := C_1 C_2.$$

Now we fix $0 < t < L(R)^{-1}$. Given $0 < \gamma \leq \alpha$, there are $s, r \in (0, R)$ such that

$$t = L(s)^{-1} \quad \text{and} \quad \gamma t = L(r)^{-1}.$$

Then $U_{\gamma t} = V_r \subset V_{s/2}$, by (4.7). So, by Proposition 4.2,3, for every $x \in U_{\gamma t}$,

$$\mu_x^{U_{\gamma t}}(U_t^c) = \mu_x^{V_r}(V_s^c) = \mathbb{P}^x[X_{\tau_r} \notin V_s] \leq C_1 C_2 L(s)/L(r) = C_1 C_2 \gamma.$$

So (J₂) holds.

To prove (J₁) let $\gamma = \alpha$. By (4.7), $U_{\alpha^2 t} \subset V_r$ (consider αt instead of t). Finally, suppose that A is a universally measurable set in $S := U_t \setminus U_{\alpha t} = V_s \setminus V_r$ and let μ be as in Proposition 4.2,1. Then there exists a closed set F contained in A or in $S \setminus A$ such that $\mu(F) > \mu(S)/3$. Since $L(s) = \alpha L(r)$, Proposition 4.2 shows that, for every $x \in U_{\alpha^2 t} \subset V_{r/2}$,

$$\mu_x^{U_{\alpha^2 t}}(F) = \mathbb{P}^x[X_{\tau_r} \in F] \geq (C_1 C_3)^{-1} \frac{\ln(\alpha^{-1})}{\alpha^{-1}} \frac{\mu(F)}{\mu(S)} > \delta_0.$$

□

Proof of Theorem 4.1. Let $0 < r < R$, $x \in V_r$, and $h \in \mathcal{H}_b(B_r)$. By Corollary 4.6 and Theorem 2.3,

$$|h(x) - h(x_0)| \leq C \|h\|_{\infty} (\rho_0(x)/\tilde{r})^{\beta} = C \|h\|_{\infty} L(r)^{\beta} L(|x - x_0|)^{-\beta}.$$

□

COROLLARY 4.7. *Let $0 < r < R$. Then, for all $h \in \mathcal{H}_b(B_r)$ and $x, y \in B_{r/3}$,*

$$(4.8) \quad |h(x) - h(y)| \leq c_1 C \|h\|_{\infty} L(r)^{\beta} L(|x - y|)^{-\beta}.$$

Proof. Let $x, y \in B_{r/3}$. Then $x \in B_{2r/3}(y) \subset B_r$, and hence, by Theorem 4.1,

$$|h(x) - h(y)| \leq C \|h\|_{\infty} L(2r/3)^{\beta} L(|x - y|)^{-\beta}.$$

where $L(2r/3) \leq L(r/2) \leq c_1 L(r)$.

□

5 Examples

Our assumptions in Section 4 are satisfied under the main assumptions made in [14], and hence Corollary 4.7 implies the statement of [14, Theorem 3] in the case $f = 0$.

Indeed, our (K) and (L₀) are localized versions of (A₁) and (K₀) (which, incidentally, imply (l₁)). The second inequality in (L₁) amounts to (l₃) (with R in place of R_0). Property (l₂) means that $l(v)/l(u) \geq c_L(v/u)^{-\gamma}$ for all $0 < u \leq v < R_0$. In particular, it leads to $l(r/2) \leq 2^{\gamma} c_L^{-1} l(r)$ for all $0 < r < R_0$. Moreover, it also implies that, for some $c > 0$,

$$(5.1) \quad \int_0^r sl(u) du \leq cr^2 L(r), \quad 0 < r < R,$$

a fact which is part of the proof of [14, (8)] without having been stated separately. Indeed, suppose that (l_2) holds, let $a := 1 - (R/R_0)^\gamma$ (so that $a = 1$ if $R_0 = \infty$) and $0 < r < R$. Then (see the proof of [14, Lemma 7]),

$$\begin{aligned} \frac{L(r)}{l(r)} &= \int_r^{R_0} u^{-1} \frac{l(u)}{l(r)} du \geq c_L r^\gamma \int_r^{R_0} u^{-1-\gamma} du \\ &= \gamma^{-1} c_L (1 - (r/R_0)^\gamma) \geq \gamma^{-1} c_L a. \end{aligned}$$

Further (see the proof of [14, (8)]),

$$\frac{1}{l(r)} \int_0^r ul(u) du \leq c_L^{-1} r^\gamma \int_0^r u^{1-\gamma} du = (2 - \gamma)^{-1} c_L^{-1} r^2.$$

Thus (5.1) holds with $c := a^{-1} c_L^{-2} \gamma / (2 - \gamma)$.

Finally, for given l and R , (L_3) is no problem.

Our assumptions are satisfied as well in the second part of [14] (beginning with Section 5), where l is assumed to be locally bounded and to vary regularly at zero with index $-\alpha \in (-2, 0]$. Implicitly, this has already been used in Section 6 of that paper and based on considerations in its Appendix. Thus Corollary 4.7 also implies [14, Theorem 12].

We note, however, that in the case $l(u) = u^{-2}(\ln u^{-1})^{-2}$ property (L_2) does not hold, since an easy calculation shows that $L \approx l$, whereas $r^{-2} \int_0^r ul(u) du \approx r^{-2}(\ln r^{-1})^{-1} = l(r) \ln r^{-1}$.

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